Strong Constructive Mathematics

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Bishop 1970

... the numerical content of intuitionistic mathematics is diluted by over-reliance on negativistic techniques. The role of negation in predictive mathematics is philosophically secure, if only because there exist negative statements that do have numerical content. Nevertheless, it is remarkable that a systematic effort to avoid negation leads uniformly to better results.

- Strong Constructive Mathematic (SCM) is the subsystem of BISH developed by avoiding weak negation (and employing strong negation and other strong concepts). Within SCM many classical methods can be understood, or translated, constructively.
- Theories developed, or under development, within SCM: Bishop-Cheng Measure Theory, theory of sets with an inequality, theory of complemented subsets, swap algebras and swap rings, complemented topology.

The avoidance of weak negation goes back to Griss.

The strong concept of an inequality on a set goes back to Brouwer.

Inequality on a set $(X, =_X)$

$(\mathtt{Ineq}_1)$ $\forall_{x,y \in X}\big(x =_X y \ \& \ x \neq_X y \Rightarrow \bot\big)$, where $\bot := 0 =_{\mathbb{N}} 1$,

$(\mathtt{Ineq}_2)$ $\forall_{x,x',y,y' \in X}\big(x =_X x' \ \& \ y =_X y' \ \& \ x \neq_X y \Rightarrow x' \neq_X y'\big)$,

$(\mathtt{Ineq}_3)$ $\forall_{x,y \in X}\big(\neg(x \neq_X y) \Rightarrow x =_X y\big)$,

$(\mathtt{Ineq}_4)$ $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow y \neq_X x\big)$,

$(\mathtt{Ineq}_5)$ $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow \forall_{z \in X}(z \neq_X x \ \vee \ z \neq_X y)\big)$,

$(\mathtt{Ineq}_6)$ $\forall_{x,y \in X}\big(x =_X y \vee x \neq_X y\big)$.

An apartness relation $(\mathtt{Ineq}_1, \mathtt{Ineq}_4, \mathtt{Ineq}_5)$ is extensional $(\mathtt{Ineq}_2)$.

We call $\mathcal{X} := (X, =_X, \neq_X)$ a set with an inequality. If $\mathcal{Y} := (Y, =_Y, \neq_Y)$ is a set with inequality, a function $f : X \to Y$ is strongly extensional, if

$$f(x) \neq_Y f(x') \Rightarrow x \neq_X x'.$$

A definition of Coquand-Palmgren in [7]

A commutative ring with a stong apartness relation is a structure $\boldsymbol{X} := (X, =_X, \neq_X; +, \cdot, 0)$, such that, besides $(\mathtt{Ineq}_0)$, $(\mathtt{Ineq}_3)$ and $(\mathtt{Ineq}_4)$, the following axioms are satisfied:

$(\mathtt{Ineq}_7)$ $\forall_{x,y,z \in X}\big(x \neq_X y \Rightarrow x + z \neq_X y + z\big)$,

$(\mathtt{Ineq}_8)$ $\forall_{x,y \in X}\big(x \cdot y \neq_X 0 \Rightarrow x \neq_X 0 \ \& \ y \neq_X 0\big)$,

$(\mathtt{Ineq}_9)$ $\forall_{x,y \in X}\big(x + y \neq_X 0 \Rightarrow x \neq_X 0 \vee y \neq_X 0\big)$.

If X has a unit, then one can also add the axiom $0 \neq_X 1$.

Proposition

Let $\boldsymbol{X} := (X, =_X, \neq_X; +, \cdot, 0)$ *be a commutative ring, such that* $\neq_X$ *satisfies* $(\mathtt{Ineq}_0), (\mathtt{Ineq}_2)$ *and* $(\mathtt{Ineq}_7)$*. The following hold:*

(i) *Strong extensionality of* $+$ *is equivalent to* $(\mathtt{Ineq}_9)$*.*

(ii) *Strong extensionality of* $\cdot$ *implies* $(\mathtt{Ineq}_8)$*.*

(iii) $(\mathtt{Ineq}_8)$ *and* $(\mathtt{Ineq}_9)$ *imply strong extensionality of* $\cdot$*.*

(iv) *Strong extensionality of* $\cdot$ *and* $+$ *is equivalent to* $(\mathtt{Ineq}_8)$ *and* $(\mathtt{Ineq}_9)$*.*

(v) *An extensional, strong apartness relation on* $\boldsymbol{X}$ *is an apartness relation i.e.,* $(\mathtt{Ineq}_5)$ *holds.*

Constructive logic with strong negation (CLSN)

- $\neg(A \wedge B) \nrightarrow \neg A \vee \neg B$ and $\neg\big(\forall_x \phi(x)\big) \nrightarrow \exists_x \neg\phi(x)$.
- In analogy to strong $\vee, \exists$, Nelson [13] and Markov [11], introduced strong negation $\sim A$ of A.
- Following Kleene's recursive realisability, Nelson developed constructive arithmetic with $\sim A$ and showed that it has the same expressive power with HA.
- The equivalences $\sim(A \wedge B) \Leftrightarrow \sim A \vee \sim B$ and $\sim\forall_x A(x) \Leftrightarrow \exists_x \sim A(x)$ are Nelson-realisable.
- In axiomatic CLSN [19, 9] these equivalences are axioms.
- In most formalisations of CLSN: $\sim A \Rightarrow \neg A$.
- Markov [11] expressed weak negation through strong negation and implication: $\neg A := A \rightarrow \sim A$.
- Rasiowa [19] introduced strong implication $A \xrightarrow{s} B$, defined by

$$A \xrightarrow{s} B := (A \rightarrow B) \wedge (\sim B \rightarrow \sim A).$$

- Model theory of CLSN in [9, 1, 21, 22].

Shulman: Affine logic for CM, 2022

He showed that numerous concepts of CM arise automatically from an “antithesis” translation of affine logic into intuitionisitic logic (IL) via a Chu/Dialectica construction.

In [18] we do something similar, but

we work within BISH, we define strong negation and we also use Rasiowa's strong implication.

In [10] we extend the definition of strong negation to inductively and coinductively defined predicates of TCF.

It is not clear how this can be done in MLTT and HoTT, but it might be possible in H.O.T.T.

Formulas in BISH

Prime formulas:

$s =_{\mathbb{N}} t$, $s \neq_{\mathbb{N}} t$, where s, t are elements of $\mathbb{N}$.

Complex formulas:

If A, B are formulas, then $A \vee B, A \wedge B, A \Rightarrow B$ are formulas.

If S is a set and $\phi(x)$ is a formula, for every variable x of set S, then $\exists_{x \in S}\big(\phi(x)\big)$ and $\forall_{x \in S}\big(\phi(x)\big)$ are formulas.

This definition of formulas is also found in a formal system of Bishop in his unpublished work.

Contrary to ZF and CZF, and in complete analogy to MLTT, the expressions $x \in X, a \in X$ are not formulas.

The quantified formulas of BISH correspond to the Sigma- and Pi-type of MLTT i.e., if $S : \mathcal{U}$ and $\phi \colon S \to \mathcal{U}$ is a type family over S, we have the analogies:

$$\forall_{x \in S} \phi(x) \ldots\ldots\ldots\ldots\ldots\ldots \prod_{x : S} \phi(x)$$

$$\exists_{x \in S} \phi(x) \ldots\ldots\ldots\ldots\ldots\ldots \sum_{x : S} \phi(x)$$

Weak negation in BISH

$$\neg A := A \Rightarrow \bot,$$

$$\bot := 0 =_{\mathbb{N}} 1$$

$$\top := 0 \neq_{\mathbb{N}} 1$$

Strong negation in BISH

$$(s =_{\mathbb{N}} t)^{\mathbf{N}} := s \neq_{\mathbb{N}} t$$

$$(s \neq_{\mathbb{N}} t)^{\mathbf{N}} := s =_{\mathbb{N}} t.$$

$$(A \vee B)^{\mathbf{N}} := A^{\mathbf{N}} \wedge B^{\mathbf{N}}$$

$$(A \wedge B)^{\mathbf{N}} := A^{\mathbf{N}} \vee B^{\mathbf{N}}$$

$$(A \Rightarrow B)^{\mathbf{N}} := A \wedge B^{\mathbf{N}}$$

$$\left(\exists_{x \in S} \phi(x)\right)^{\mathbf{N}} := \forall_{x \in S} \left(\phi(x)^{\mathbf{N}}\right)$$

$$\left(\forall_{x \in S} \phi(x)\right)^{\mathbf{N}} := \exists_{x \in S} \left(\phi(x)^{\mathbf{N}}\right)$$

Proposition

Let A be a formula of BISH.

(i) $A^{\mathbf{NN}} \Rightarrow A$.

(ii) $A^{\mathbf{N}} \Rightarrow \neg A$.

(iii) $A \wedge A^{\mathbf{N}} \Rightarrow \bot$.

$$(\neg A)^{\mathbf{N}} := (A \Rightarrow 0 =_{\mathbb{N}} 1)^{\mathbf{N}} := A \wedge 0 \neq_{\mathbb{N}} 1 \Leftrightarrow A$$

$$(A \Rightarrow A)^{\mathbf{N}} := A \wedge A^{\mathbf{N}}$$

$$(A \Rightarrow A)^{\mathbf{NN}} := \left(A \wedge A^{\mathbf{N}}\right)^{\mathbf{N}} := A^{\mathbf{N}} \vee A^{\mathbf{NN}} \Rightarrow \neg A \vee A$$

Hence, we cannot accept, in general, $A \Rightarrow A^{\mathbf{NN}}$.

The strong inequality of a defined set $(X, =_X)$:

$$\left(x =_X x'\right)^{\mathbf{N}}.$$

We call $(X, =_X, =_X^{\mathbf{N}})$ a set with its strong inequality.

We call $(X, =_X, \neq_X)$ a a set with a strong inequality, and $\neq_X$ is called strong, if

$$x \neq_X x' \Leftrightarrow (x =_X x')^{\mathbf{N}}.$$

The strong inequality is strongly tight:

$$\left(x =_X x'\right)^{\mathbf{NN}} \Rightarrow x =_X x'$$

Richman: to define an inequality for every set would be "cumbersome and easily forgotten".

In most cases, but not all, the inequalities considered are strong!

$\neq_{\mathbb{R}}$ is strong

$$x =_{\mathbb{R}} y :\Leftrightarrow \forall_{n \in \mathbb{N}^+} \left(|x_n - y_n| \leqslant \frac{2}{n} \right)$$

$$\begin{aligned}
(x =_{\mathbb{R}} \; y)^{\mathbf{N}} &:\Leftrightarrow \exists_{n \in \mathbb{N}^+} \left(\left(|x_n - y_n| \leqslant \frac{2}{n} \right)^{\mathbf{N}} \right) \\
&\Leftrightarrow \exists_{n \in \mathbb{N}^+} \left(|x_n - y_n| > \frac{2}{n} \right) \\
&\Leftrightarrow |x - y| > 0 \\
&\Leftrightarrow: x \neq_{\mathbb{R}} y
\end{aligned}$$

The strong inequality of the product set

$$\left[(x,y) =_{X\times Y} (x',y')\right]^{\mathbf{N}} := \left[x =_X x' \wedge y =_Y y'\right]^{\mathbf{N}}$$
$$:= (x =_X x')^{\mathbf{N}} \vee (y =_Y y')^{\mathbf{N}}.$$

In the general case we define

$$(x,y) \neq_{X\times Y} (x',y') :\Leftrightarrow x \neq_X x' \vee y \neq_Y y'$$

The strong inequality of the function set

$$\left[f =_{\mathbb{F}(X,Y)} g\right]^{\mathbf{N}} := \left[\forall_{x\in X}\left(f(x) =_Y g(x)\right)\right]^{\mathbf{N}}$$
$$:= \exists_{x\in X}\left[f(x) =_Y g(x)\right]^{\mathbf{N}}$$

In the general case we define

$$f \neq_{\mathbb{F}(X,Y)} g :\Leftrightarrow \exists_{x\in X}\left(f(x) \neq_Y g(x)\right)$$

Ther strong inequality need not be an apartness relation

$$
\begin{aligned}
(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) &:\Leftrightarrow i =_I j \;\&\; \lambda_{ij}(x) =_{\lambda_0(j)} y \\
&:\Leftrightarrow i =_I j \;\&\; [i =_I j \Rightarrow \lambda_{ij}(x) =_{\lambda_0(j)} y]
\end{aligned}
$$

$$
[(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y)]^{\mathbf{N}} :\Leftrightarrow (i =_I j)^{\mathbf{N}} \vee \big(i =_I j \wedge [\lambda_{ij}(x) =_{\lambda_0(j)} y]^{\mathbf{N}}\big)
$$

Even if $=_I^{\mathbf{N}}$ and $=_{\lambda_0(j)}^{\mathbf{N}}$ are apartness relations, I needs to be discrete.

Rasiowa's strong implication in BISH

$$A \overset{s}{\Rightarrow} B := (A \Rightarrow B) \wedge \left(B^{\mathbf{N}} \Rightarrow A^{\mathbf{N}}\right)$$

$$a \leqslant 0 \wedge b \leqslant 0 \overset{s}{\Rightarrow} a + b \leqslant 0$$

$$a \leqslant 0 \wedge b \leqslant 0 \Rightarrow a + b \leqslant 0$$

$$a + b > 0 \Rightarrow a > 0 \vee b > 0$$

The implication

$$a \neq_{\mathbb{R}} 0 \wedge b \neq_{\mathbb{R}} 0 \Rightarrow a \cdot b \neq_{\mathbb{R}} 0$$

is not strong in BISH; if it was, the implication

$$a \cdot b =_{\mathbb{R}} 0 \Rightarrow a =_{\mathbb{R}} 0 \vee b =_{\mathbb{R}} 0$$

would be derivable, which is equivalent to LLPO.

Open question:

Which implications in the theory of

$$(\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}}, +, \cdot, <, \leqslant)$$

are strong?

Strong functions

An a.r. $f\colon (X, =_X) \rightsquigarrow (Y, =_Y)$ is a function, if

$$x =_X x' \Rightarrow f(x) =_Y f(x').$$

A function $f\colon (X, =_X, =^{\mathbf{N}}_X) \to (Y, =_Y, =^{\mathbf{N}}_Y)$ is strong, if

$$[f(x) =_Y f(x')]^{\mathbf{N}} \Rightarrow (x =_X x')^{\mathbf{N}}$$

i.e., if

$$x =_X x' \overset{s}{\Rightarrow} f(x) =_Y f(x').$$

Proposition

Let $\mathcal{X} := (X, =_X, \neq_X)$ and $\mathcal{Y} := (Y, =_Y, \neq_Y)$ such that $\neq_X$ and $\neq_Y$ are strong. Then a function $f\colon X \to Y$ is strong if and only if it is strongly extensional.

Let X be a totality and $d \colon X \times X \rightsquigarrow [0, +\infty)]$ an assignment routine. If

$$x =_X y :\Leftrightarrow d(x, y) =_{\mathbb{R}} 0,$$

its strong inequality is

$$(x =_X y)^{\mathbf{N}} :\Leftrightarrow [d(x, y) =_{\mathbb{R}} 0]^{\mathbf{N}} \Leftrightarrow d(x, y) \neq_{\mathbb{R}} 0 \Leftrightarrow d(x, y) > 0.$$

Proposition

If $d(x, x) =_{\mathbb{R}} 0$, for every $x \in X$, and if d is symmetric and satisfies the triangle inequality, then $=_X$ is an equality on X and d is a strong function.

In BISH we cannot accept that all functions are strong

The following are equivalent:

(i) Markov's principle.

(ii) Every function $f\colon (\mathbb{R}, =_{\mathbb{R}}, =^{\mathbf{N}}_{\mathbb{R}}) \to (\mathbb{R}, =_{\mathbb{R}}, =^{\mathbf{N}}_{\mathbb{R}})$ is strong.

(iii) $\neg(x =_{\mathbb{R}} 0) \Rightarrow (x =_{\mathbb{R}} 0)^{\mathbf{N}}$.

(iv) $\neg(x \leqslant y) \Rightarrow x > y$.

Hence, in BISH we cannot accept:

$$(A \Rightarrow B) \Rightarrow \big(B^{\mathbf{N}} \Rightarrow A^{\mathbf{N}}\big),$$

$$\neg A \Rightarrow A^{\mathbf{N}},$$

$$\neg\big(A^{\mathbf{N}}\big) \Rightarrow A,$$

as $(x > y)^{\mathbf{N}} \Leftrightarrow x \leqslant y$.

Strong empty subset

If $(X, =_X)$ is a set and $=_X^{\mathbf{N}}$ is extensional, its strong empty subset is

$$\square_X^{\mathbf{N}} := \{x \in X \mid (x =_X x)^{\mathbf{N}}\}.$$

In general, if $\neq_X$ is an inequality on X the $\neq_X$-empty subset of X is

$$\not\square_X := \{x \in X \mid x \neq_X x\}.$$

One can prove many properties of $\not\square_X$ in MIN.

Bishop defined in a negative way the (categorical) empty subset of an inhabited subset of X; and not the empty set.

Bridges defined the empty subset negativistically and using the judgment $x \in X$ as a formula

$$\emptyset_X := \{x \in X \mid \neg(x \in X)\}.$$

Strong complement

$\mathcal{X} := (X, =_X, \neq_X)$, where $=_X^{\mathbf{N}}$ and $\neq_X$ are extensional.
If $C \subseteq X$ extensional i.e., $C := \{x \in X \mid P(x)\}$ and $P(x)$ is extensional on X, then

$$C =_{\mathcal{E}(X)} \{x \in X \mid \exists_{c \in C}(x =_X c)\}.$$

Its strong complement is

$$C^{\mathbf{N}} := \Big\{x \in X \mid \forall_{c \in C}\big((c =_X x)^{\mathbf{N}}\big)\Big\}.$$

In the general case it is natural to define its $\neq_X$-complement

$$C^{\neq_X} := \Big\{x \in X \mid \forall_{c \in C}(c \neq_X x)\Big\}.$$

Clearly, if $\neq_X$ is strong, then $C^{\mathbf{N}} =_{\mathcal{E}(X)} C^{\neq_X}$.

Intersection and disjointness

If $C, D \subseteq X$ extensional, then they intersect, or they ovelap, if

$$C \between D :\Leftrightarrow \exists_{c \in C} \exists_{d \in D} (c =_X d)$$

and they are strongly disjoint, if

$$C)(_{\mathbf{N}} D :\Leftrightarrow \forall_{c \in C} \forall_{d \in D} \big((c =_X d)^{\mathbf{N}} \big).$$

In the geberal case, we define C, D to be $\neq_X$-disjoint, if

$$C)(_{\neq_X} D :\Leftrightarrow \forall_{c \in C} \forall_{d \in D} (c \neq_X d).$$

If

$$C^{\mathbf{N}} := \{x \in X \mid P(x)^{\mathbf{N}}\},$$

then we cannot show that C and $C^{\mathbf{N}}$ are (strongly) disjoint.

Tight formulas and tight concepts

$A^{\mathbf{N}}$ is tight if and only if

$$\neg\left(A^{\mathbf{N}}\right) \Rightarrow A$$

If $C \subseteq X$, we call $C^{\mathbf{N}}$ tight, if

$$\left(C^{\mathbf{N}}\right)' \subseteq C.$$

Proposition

If C is a closed and located subset of a metric space, then $C^{\mathbf{N}}$ is tight.

The proof avoids countable choice!

Definition

A strong set is a totality X equipped with a strong equivalence relation $x =_X x'$ i.e., the implications involved in the definitional clauses of an equivalence relation are satisfied in the strong sense.

(se_1) $\forall_{x \in X}(x =_X x)$.

(se_2) $\forall_{x,y \in X}\big[\big(x =_X y \Rightarrow y =_X x\big) \;\&\; (y =_X x)^{\mathbf{N}} \Rightarrow (x =_X y)^{\mathbf{N}}\big)\big]$.

(se_3) $\forall_{x,y,z \in X}\big[\big(x =_X y \;\&\; y =_X z \Rightarrow x =_X z\big) \;\&\; (x =_X z)^{\mathbf{N}} \Rightarrow (x =_X y)^{\mathbf{N}} \vee (y =_X z)^{\mathbf{N}}\big]$.

Example

The primitive set of natural numbers $(\mathbb{N}, =_{\mathbb{N}})$ is strong, as by definition of strong negation $(n =_{\mathbb{N}} m)^{\mathbf{N}} := n \neq_{\mathbb{N}} m$.

Example

Let $*$ be a constant in the language of BISH, like the constant 0. The totality $\mathbb{1} := \{*\}$ is defined by the membership condition $z \in \mathbb{1} :\Leftrightarrow z := *$, and its equality is defined by the formula $z =_{\mathbb{1}} w :\Leftrightarrow \top$. Consequently, $(z =_{\mathbb{1}} w)^{\mathbf{N}} :\Leftrightarrow \bot$, and the unit set $(\mathbb{1}, =_{\mathbb{1}})$, is strong, as (se_2) is reduced to $\bot \Rightarrow \bot$ and (se_2) is reduced to $\bot \Rightarrow \bot \vee \bot$. I.e., this proof is within MIN.

Proposition

(i) (se_3) *implies the extensionality of* $=_X^{\mathbf{N}}$

(ii) *If* $(X, =_X)$ *is a strong set, then the relation* $(x =_X y)^{\mathbf{N}}$ *is an apartness relation.*

Definition

Let **StrongSet** be the category of strong sets and strong functions. A (strong) function $f\colon X \to Y$ is called a strong embedding, if it satisfies the strong implication

$$f(x) =_Y f(x') \overset{s}{\Rightarrow} x =_X x',$$

for every $x, x' \in X$, i.e.,

$$f(x) =_Y f(x') \Rightarrow x =_X x'$$

and

$$(x =_X x')^{\mathbf{N}} \Rightarrow (f(x) =_Y f(x'))^{\mathbf{N}},$$

i.e., f is an embedding and an injection w.r.t. $=_X^{\mathbf{N}}$.

StrongSet is a full subcategory of **SetStrongIneq**.

Proposition

The category **StrongSet** *is the subcategory of* **SetStrongIneq** *whith objects sets with an inequality, which is equivalent to strong negation of equality and it is an apartness relation.*

Proof.

If $(X, =_X)$ is strong, then its inequality $=_X^{\mathbf{N}}$ is an apartness relation. Converely, if $\mathcal{X} := (X, =_X, \neq_X)$ is in **SetStrongIneq**, then for every $x, x' \in X$ we have that $x \neq_X x' \Leftrightarrow (x =_X x')^{\mathbf{N}}$. To show (se_2), we suppose $(x =_X x')^{\mathbf{N}}$, which is equivalent to $x \neq_X x'$, hence by $(\mathtt{Ineq}_4)$ we get $'x \neq_X x$, which in turn implies $(x' =_X x)^{\mathbf{N}}$. To show (se_3), we suppose $(x =_X z)^{\mathbf{N}}$, which is equivalent to $x \neq_X z$, hence by $(\mathtt{Ineq}_5)$ we get $x \neq_X y$ or $y \neq_X z$. In the first case we get $(x =_X y)^{\mathbf{N}}$, while in the second we get $(y =_X z)^{\mathbf{N}}$. □

Corollary

The set $\mathbb{R}$ of real numbers is strong.

Proof.

The structure $\mathcal{R} := (\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}})$ is in **SetStrongIneq**, the inequality $\neq_{\mathbb{R}}$ is an apartness, and by the previous proposition we have that $(\mathbb{R}, =_{\mathbb{R}})$ is strong. $\square$

Proposition

Let $(X, =_X)$ *be a strong set,* A *a totality, and* $i_A \colon A \rightsquigarrow X$ *an assignment routine.*

(i) *If we define* $a =_A a' :\Leftrightarrow i_A(a) =_X i_A(a')$, *for every* $a, a' \in A$, *then* $(A, =_A)$ *is strong.*

(ii) *If* $P(x)$ *is an extensional property on* X, *the extensional subset* X_P *of* X *is strong.*

Proposition

Let $(X, =_X)$ *and* $(Y, =_Y)$ *be strong sets.*

(i) *The product* $(X \times Y, =_{X \times Y})$ *is strong, the projections* $\mathtt{pr}_X$ *and* $\mathtt{pr}_Y$ *are strong functions, and* $(X \times Y, =_{X \times Y})$ *is a product of* $(X, =_X)$ *and* $(Y, =_Y)$ *in* **StrongSet**.

(ii) *The function space* $(\mathbb{F}(X, Y), =_{\mathbb{F}(X,Y)})$ *is strong, and it is the exponential of* $(X, =_X)$ *and* $(Y, =_Y)$ *in* **StrongSet**.

(iii) $(\mathbb{1}, =_{\mathbb{1}})$ *is a terminal object in* **StrongSet**.

(iv) **StrongSet** *has pullbacks.*

(v) **StrongSet** *is finitely complete.*

(vi) **StrongSet** *has coproducts.*

Categories of sets and functions not visible in CLASS

$$\mathbf{StrongSet} \leqslant \mathbf{SetStrongIneq} \leqslant \mathbf{SetIneq} \leqslant \mathbf{Set}$$

$$\mathbf{StrongSet} \leqslant \mathbf{SetStrongExtIneq} \leqslant \mathbf{SetExtIneq} \leqslant \mathbf{SetIneq} \leqslant \mathbf{Set}$$

$$\mathbf{StrongSetTightIneq} \leqslant \mathbf{SetStrongTightIneq} \leqslant \mathbf{SetTightIneq} \leqslant \mathbf{SetIneq} \leqslant \mathbf{Set}$$

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