School maths for the era of AI: arithmetic, algorithmic, algebra

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The talk is based on

Borovik, Kondratiev. *A new course 'Algebra + Computer Science': What should be its outcomes and where it should start*, 2023, https://arxiv.org/abs/2212.12257 [math.HO].
Borovik, Kocsis, Kondratiev. *Mathematics and Mathematics Education in the 21st Century*, 2022, arXiv:2201.08364 [math.HO].
Borovik. *Mathematics for makers and mathematics for users*, 2017, http://bit.ly/2qYHtst.
Borovik. *Calling a spade a spade: Mathematics in the new pattern of division of labour*, 2016, http://goo.gl/TT6ncO.

Links are clickable.

Mathematics: crisis or revolution?

- Intrinsic pressures:
  - Mathematics is outgrowing the limits of human comprehension.
- External pressures:
  - Changes in the socio-economic environment.
  - Weaponisation of mathematics and computer science.
  - Emergence of AI.
  - Stagnating mathematics education.

Maths education: decline or reform?

- Socio-political pressures to downgrade mainstream maths education in schools.
- Reason:

  "This is Economy, Stupid":

  change of the role of mathematics in the division of labour in economics.
  - See **Borovik**, *Calling a spade a spade: Mathematics in the new pattern of division of labour*, 2016, http://goo.gl/TT6ncO.

Solutions

- **For research mathematics**: switch to the use of computers as interactive assistants in production of proofs.
  - But doing mathematics with interactive proof assistants requires a much better mathematics education.
- **For maths education**: split, at a relatively early stage, in
  - **Mainstream** – for the vast majority of population.
  - "**Deep Stream**", with a further subdivision at some point, into substreams
    - for future mathematicians and computers scientists
    - for non-mathematicians who will be professionally using mathematics.
- Surprisingly, there is no coherent discussion of these issues.

Timescale

The cycle of reproduction of mathematics

primary school – high school – college/university/ – return to school as a teacher

is at least 15 years; if it continues as

university/ – PhD studies – postdoc – return to university as a lecturer,

it is at least 20 years.

New research mathematics: Interactive proof assistants

Interactive proof assistants: user-friendly *interactive* theorem proving software which

- rewrites, in an interactive process, informal proofs as *formal objects*;
- checks the formal correctness of these objects.

Mathematics of the future

Universal electronic library

- of computer verifiable “proofs” and “definitions”,
- accessible to all “interactive proof assistants”, which agree with it all newly developing definitions, theorems and proofs.

Ideally, informal versions and explanations accessible to human understanding should also be produced.

Reverse analysis

How to nurture and educate mathematicians for the New Mathematics?
I approach this problem on the basis of my 45+ year of practical work.

- We cannot just wait that they will just somehow appear.
- Let us analyse the situation from the aim to the current situation, in reverse steps:
  - PhD study
  - MSc
  - BSc
  - High school
  - Middle school
  - Primary school

Answer: "Deep Stream" in school maths – as opposed to Mainstream

- Highly academically selective specialist education from age about 13 (middle school), or even earlier.
- Focus on development of **abstract thinking**.
- Surprisingly, specifics of curricula, organisation of teaching, etc. are less important in academically selective education.
  Vladimir Rokhlin:
  > *Teaching mathematics to the would-be mathematicians is infinitely easier than teaching mathematics to non-mathematicians.*
- Very significant (and diverse) experience is accumulated in countries like Hungary, France, Russia, USA, Israel, Iran.
- But in the UK and in the EU as a whole?

Links with computer science education

- Mathematics and computer science should be treated as a single subject.
- Exposure to code writing from day one.
- Some seminumerical algorithms from volume 2 of Donald Knuth's *The art of computer programming* beg to be included in the algebra lessons.
- Algebra will greatly benefit from being taught and learned in parallel with a sufficiently user-friendly programming language which includes elements of functional programming.

Abstraction

Kramer 2007: [Abstraction means] the process of removing detail to simplify and focus attention [...]:

- the act of withdrawing or removing something, and
- the act or process of leaving out of consideration one or more properties of a complex object so as to attend to others.

[It is also] the process of generalisation to identify the common core or essence [...]:

- the process of formulating general concepts by abstracting common properties of instances, and
- a general concept formed by extracting common features from specific example.

Encapsulation

Weller et al. 2004:

> *The encapsulation and de-encapsulation of process in order to perform actions is a common experience in mathematical thinking. For example, one might wish to add two functions f and g to obtain a new function $f + g$. Thinking about doing this requires that the two original functions and the resulting function are conceived as objects. The transformation is imagined by de-encapsulating back to the two underlying processes and coordinating them by thinking about all of the elements x of the domain and all of the individual transformations $f(x)$ and $g(x)$ at one time so as to obtain, by adding, the new process, which consists of transforming each x to $f(x) + g(x)$. This new process is then encapsulated to obtain the new function $f + g$.*

Encapsulation *après* Weller e.a.

To add two functions f and g to obtain a new function $f + g$:

- Treat the two original functions and the resulting function as objects.
- De-encapsulating back to the two underlying processes and coordinate them by thinking about all of the elements x of the domain and all of the individual transformations $f(x)$ and $g(x)$ at one time.
- Obtain, by adding, the new process, which consists of transforming each x to $f(x) + g(x)$.
- Encapsulated this new process to obtain the new function $f + g$.

Encapsulation / de-encapsulation and Khan Academy

Gulnar has an average score of 87 after 6 tests. What does Gulnar need to get on the next test to finish with an average of 78 on all 7 tests?

Hints provided, one after another, by the Khan Academy website:

Hint 1. Since the average score of the first 6 tests is 87, the sum of the scores of the first 6 tests is $6 \times 87 = 522$.

Hint 2. If Gulnar gets a score of x on the 7th test, then the average score on all 7 tests will be:

$$\frac{522 + x}{7}.$$

Hint 3. This average needs to be equal to 78 so:

$$\frac{522 + x}{7} = 78.$$

Hint 4 $x = 24$.

Encapsulation / de-encapsulation and "Questions method"

"Questions" method circa 1950-s:

Question 1. How many points in total did Gulnar get in 6 tests?
Answer: $6 \times 87 = 522$.

Question 2. How many points in total does Gulnar need to get in 7 tests?
Answer: $7 \times 78 = 546$.

Question 3. How many points does Gulnar need to get in the 7th test?
Answer: $546 - 522 = 24$.

These questions are **not** asked by the teacher, they are expected from **students** themselves.

Questions method

Gulnar has an average score of 87 after 6 tests. What does Gulnar need to get on the next test to finish with an average of 78 on all 7 tests?

Questions method

> *Gulnar has an average score of* 87 *after* 6 *tests. What does Gulnar need to get on the next test to finish with an average of* 78 *on all* 7 *tests?*

The teacher removes the question from the problem:

> *Gulnar has an average score of* 87 *after* 6 *tests.* . . . . . *. . . . . to finish with an average of* 78 *on all* 7 *tests.*

and asks instead:

> **What questions could be asked about these data?**

Questions method

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and asks instead:

What questions could be asked about these data?

Students are expected to **analyse** and, if needed, **de-encapsulate** the data.

Reification

Wikipedia:

> *Reification is the process by which an abstract idea about a computer program is turned into an explicit data model or other object created in a programming language.*
>
> *A computable/addressable object—a resource—is created in a system as a proxy for a non computable/addressable object.*

Reification: examples

x The notorious unknown x of elementary algebra.

$i = \sqrt{-1}$ Imaginary root of -1, the basis of complex analysis.

It took centuries for mathematicians to develop these new mathematical entities.

Is it realistic to expect from schoolchildren to do reification as a normal everyday task?

Reification in arithmetic: intermediate parameters

It takes 5 days for a steamboat to get from St Louis to New Orleans, and 7 days to return back. How long will it take for a raft to drift from St Louis to New Orleans?

Let us **introduce** a new measure of distance, called *lieue* and **set** the distance from StL to NO being equal to $5 \times 7 = 35$ lieue. Speeds of the steamboat:

$$\text{downstream } \frac{35}{5} = 7\frac{\text{lieue}}{\text{day}}, \quad \text{upstream } \frac{35}{7} = 5\frac{\text{lieue}}{\text{day}}.$$

Since the speed of the current gets added to, or subtracted from, the speed of the steamship in still water, the speed of the current is

$$\frac{7-5}{2} = 1\frac{\text{lieue}}{\text{day}}$$

and a raft will drift for $35/1 = 35$ days.

The aims of the Deep Stream

Developing in children skills of

- *identification of hidden structures in the real world, in IT, in mathematics*
- *abstraction*
- *reification and encapsulation / de-encapsulation*
- *confident handling and creation of proofs*
- *confident and fluent computer programming skills*
- *ability to create, maintain, and use mental images of big complex hierarchically built structures*

Deep Stream: psychological and cognitive traits to nurture

- *ability to engage the subconscious when doing mathematics;*
- *ability to share intuition;*
- *ability to learn by absorption;*
- *ability to compress mathematical knowledge;*
- *capacity for abstract thinking;*
- *being in control of their mathematics.*

Borovik 2017: *Mathematics for makers and mathematics for users*, http://bit.ly/2qYHtst.

What about Mainstream?

- Immense social and political pressures.
- It seems that anything goes. . .
- In some countries, mainstream mathematics education could be seriously degraded.
- It will continue to contribute to an important social role of schools:

  To serve as a storage room for children while parents are at work.

  Covid and lockdowns demonstrated the importance of this role.

Understanding of big systems

In the mid 20th century, school mathematics education contained two large complex closely knit systems of mathematical material:

- Definitions, theorems, and proofs of Euclidean geometry.
- Proofs of trigonometric identities.

Nowadays—nothing of the kind.

A new course 'Algebra + Computer Science'

Question 1 What are learning outcomes of the new course? What school graduates should be able to do on completion of the course?

Question 2 How the new course will relate to the primary school arithmetic?

'Algebra + Computer Science': Outcomes

The most important issue of the 21st Century is the relationship between people and computers.
Therefore on completing the course, Learner should

- understanding the working of AI and of dangers coming from AI saturating the economy, politics, warfare, media, everyday life;
- if necessary, be prepared to learn, and use, more technical and professional aspects of computer science and computer programming.

School graduates in the mid 20th century: were expected to master enough of algebra and trigonometry for training, if needed, as artillery officers or air pilots—it was the era of mass conscription armies.

Criteria of the mastery of the required skills:

- Learner's answer to an arithmetic or algebraic problem should be an **algorithm** implemented in computer code developed by Learner—which
  - solves *all* problems of the same type;
  - helps to check, analyse, and generalise the solution.
- This applies to so called 'real life' maths problems as well.
- **Proofs** naturally appear as proofs of validity of algorithms.

The bridge from arithmetic to algebra

The merger of school mathematics with informatics

- Should start simultaneously with the phase transition of mathematical learning from arithmetic to algebra.
- Should be rooted in those aspects of arithmetic which actually belong to computer science but are not usually recognised as such.

Arithmetic already contains

Abstraction The concept of *number* is already a huge abstraction.

Algorithms
- Long division, etc. are algorithms.
- The '**questions method**' allows children to develop their own algorithms which solve many types of arithmetic problems.

Recursion Long division, long multiplication, addition and subtraction of decimals.

Typed variables 'Named numbers' of arithmetic perfectly map to typed variables of computer programming.

Reification Examples were given.

For more:

Read *Executive summary* in

Borovik, Kondratiev. *A new course 'Algebra + Computer Science': What should be its outcomes and where it should start*, 2023, https://arxiv.org/abs/2212.12257 [math.HO].

Thanks for your attention!