On the Necessity of Some Topological Spaces

Robert Lubarsky
Florida Atlantic University

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Topological Semantics for Constructive Logic

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$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$

$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$

$[\![\neg\phi]\!] = \text{int}(\mathcal{T} \setminus [\![\phi]\!])$

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$[\![\phi \rightarrow \psi]\!] = \bigcup\{\mathcal{O} \mid \mathcal{O} \cap [\![\phi]\!] \subseteq [\![\psi]\!]\}$
Notation: $\mathcal{O} \subseteq [\![\phi]\!]$ iff $\mathcal{O} \leq [\![\phi]\!]$ iff $\mathcal{O} \Vdash \phi$

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Theorem

$\mathcal{T} \not\Vdash$ LEM iff Sierpinski space is a quotient space of $\mathcal{T}$.

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Theorem

$\mathcal{T} \not\Vdash$ WLEM iff three-point Sierpinski space is a quotient space of $\mathcal{T}$.

Example: Getting Extensions of Weak Excluded Middle to Fail

WLEM_ω: For any countable sequence of propositions, if it is impossible that for any distinct pair of them both are true, then one of them must be false: $\neg[\bigvee_{i,j\in\omega,i\neq j}(A_i \wedge A_j)] \rightarrow \bigvee_i \neg A_i$.

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Theorem

$\mathcal{T} \not\Vdash$ WLEM$_\omega$ iff ω^+ with the coarse topology is a quotient space of $\mathcal{T}$.

LPO and LLPO

LPO, the Limited Principle of Omniscience: Every binary sequence is either all 0s or has a 1 in it.
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Let $\mathcal{U}$ be a non-principal ultrafilter on ω. The $\mathcal{U}$-induced ultrafilter topology on ω is discrete on ω, and the open sets containing * are exactly sets of the form $\{*\} \cup u$, where $u \in \mathcal{U}$.

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The ultrafilter topology forces LLPO and not LPO.

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Theorem

If $x \in \mathcal{T}$ forces LLPO and does not force LPO, then a subspace of $\mathcal{T}$ has a quotient space which is an ultrafilter topology on ω.

(To say x forces ϕ means some neighborhood of x forces ϕ, and that x does not force ϕ means no neighborhood of x forces ϕ.)

MP and WMP

MP, Markov's Principle: If it is impossible for every value in a binary sequence to be 0, then there is one value which is 1.
WMP, Weak Markov's Principle:

$$\forall\gamma\ [\forall\beta\ (\neg\neg\exists n\ (\beta(n) = 1) \vee \neg\neg\exists n\ (\gamma(n) = 1 \wedge \beta(n) = 0)) \to \exists n\ \gamma(n) = 1].$$

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Theorem

If $x \in \mathcal{T}$ forces WMP and does not force MP, then it is dense at x that there is a quotient space which is a non-principal non-ultra filter topology on ω.

(To say that a property P holds densely at x means every open set containing x has an open subset satisfying P.)

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Question: When does $\mathcal{T}$ force there to be a new real?

When Is There a New Real?

Definition

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Theorem

Every topological space has a canonical, universal, functionally Hausdorff quotient space.

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Theorem

If $\mathcal{T}$ is functionally Hausdorff and separable, then $\mathcal{T} \Vdash$ "There is a real which is not in the ground model." iff points in $\mathcal{T}$ are not open.

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Definition

$\mathcal{T}$ has **local functional covering dimension 0** if, for every continuous $g : \mathcal{T} \to \mathbb{R}$ there is an open cover $\{\mathcal{O}_j \mid j \in J\}$ of $\mathcal{T}$ such that, for every $j \in J$ and open cover $\{\mathcal{U}_i \mid i \in I\}$ of $\mathbb{R}$, $\{g^{-1}(\mathcal{U}_i) \mid i \in I\}$ has a refinement of disjoint clopens which covers $\mathcal{O}_j$.

Results

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Theorem

$\mathcal{T}$ forces there is a real which is not a regular Cauchy real iff $\mathcal{T}$ has an open cover of which each member $\mathcal{O}$ has a hereditary witness that $\mathcal{O}$ is not of local functional covering dimension 0.

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- Some logical principles are, under Dependent Choice, equivalent with principles of the reals. For instance, under DC, LPO is equivalent with the decidability of equality on $\mathbb{R}$. How could the difference between the logical and analytical versions be characterized topologically?

Questions

- How can you characterize the failure, or the satisfaction, of other principles of analysis, such as BD-N?
- How could the difference between the logical and analytical versions of principles be characterized topologically?
- How can we refine these results? For instance, we extracted the coarse topology on ω^+ from $\neg\text{WLEM}_\omega$, but actually this topology also satisfies LPO. What if we assume not only $\neg\text{WLEM}_\omega$ but also LPO? Similarly, we got the ultrafilter topology by assuming LLPO + $\neg$LPO; actually, in that model, WLEM holds and WMP fails. What more could we conclude about the topology from WLEM + $\neg$WMP?