De Morgan's law and related principles

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What is reverse mathematics?

Something often in usual mathematics

- From the assumption A, the conclusion B is derived.
- (Weaken A if possible)
- Sometimes, the conclusion B also derive the assumption A
  (A and B are equivalent)

- If we take a set comprehension axiom as A, then we can classify usual theorems in mathematics by set comprehension axioms.

Friedman-Simpson's reverse math

- Reverse mathematics with classical logic, using systems of second order arithmetic
- "The reverse mathematics" (?)

Language of 2nd order arithmetic and the system RCA_0

Language L_2

- Constants $0, 1$
- Binary functions $+$, $\cdot$
- Binary relations $<$, $\in$
- 1st order variables x, y, z,...
- 2nd order variables X, Y, Z,...
- 1st order equality $=$

The system RCA_0

Basic arithmetic

Successor $n + 1 \neq 0, \quad n + 1 = m + 1 \to n = m,$

Addition $n + 0 = n, \quad n + (m + 1) = (n + m) + 1,$

Multiplication $n \cdot 0 = 0, \quad n \cdot (m + 1) = n \cdot m + n,$

Order $\neg m < 0, \quad m < n + 1 \leftrightarrow m \leq n,$

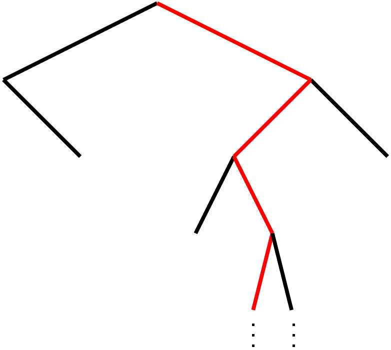
Σ^0_1 induction $A(0) \wedge \forall n (A(n) \to A(n+1)) \to \forall n A(n)$, for $A \in \Sigma^0_1$.

Δ^0_1 comprehension $\forall n (A(n) \leftrightarrow B(n)) \to \exists X \forall n (A(n) \leftrightarrow n \in X)$,

for $A \in \Sigma^0_1$ and $B \in \Pi^0_1$.

Weak König's lemma

- A binary *tree* T is a subset of $\{0, 1\}^*$ closed under initial segments.
- A binary tree T is *infinite* if $\forall n \exists s \in T(|s| = n)$, where $|s|$ is the length of a finite tree s.
- A *path* of a binary tree T is a function α s.t. $\forall n(\overline{\alpha}n \in T)$, where $\overline{\alpha}n$ is a finite sequence $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$.
- *Weak König's Lemma (*WKL*)*: "Every infinite binary tree has a path"



Some results from Friedman-Simpson reverse math

TFAE over RCA_0

- Weak König's lemma: Every infinite binary tree has a path.
- Heine-Borel's covering theorem
- Every continuous function on $[0,1]$ is uniformly continuous.
- Every continuous function on $[0,1]$ has infimum.
- Every continuous function on $[0,1]$ has a point attaining the infimum.
- Every continuous function on $[0,1]$ is Riemann integrable.
- Gödel's completeness theorem
- Every countable ring contains a prime ideal.
- Brouwer's fixed point theorem
- Peano's existence theorem for solution of ODE.
- Separable Hahn-Banach theorem
- Π^0_1 axiom of choice

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$Weak König's lemma

Intuitionistic logic and constructive reverse math

Usual mathematics

Based on classical logic

Constructive mathematics

Based on intuitionistic logic

Constructive reverse mathematics

A mathematical theorem are characterized with a combination of

- choice principle (asserting the existence of a function)
- logical principles

which are necessary and sufficient to prove it.

Base theory EL_0

Language L_{EL}

- Constant 0
- Function symbols for all primitive recursive functions S, f,.....
- Application symbol AP
- Abstraction operator λ
- recursor $\mathbf{r}$
- 1st order $=$
- 1st order variables x, y, z,...
- 2nd order variables α, β, γ,...

System EL_0

Successor $\neg S0 = 0$

Defining equations for primitive recursive functions

$$x + 0 = x,\ x + Sy = S(x + y);...$$

Π^0_0 induction $A(0) \wedge \forall n(A(n) \rightarrow A(n+1)) = \forall n A(n)$, for $A \in \Pi^0_0$

λ conversion $(\lambda x.t)s = t[x/s]$

Primitive recursion $\mathbf{r}(t, \varphi, 0) = t,\ \mathbf{r}(t, \varphi, St') = \varphi(\mathbf{r}(t, \varphi, t'), t')$

$\mathrm{QF\text{-}AC}^{00}$ $\forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(y))$, for $A(x, y) \in \Pi^0_0$

RCA_0 and EL_0

RCA_0

- Classical logic
- Set based language
- Allowing primitive recursion

EL_0

- Intuitionistic logic
- Function based language
  - It yields $A \vee \neg A$ for Π^0_0 formulae
- Allowing primitive recursion

Fact ([6])

For any Π^0_2 sentence A in L_2, $\mathsf{RCA}_0 \vdash A$ yields $\mathsf{EL}_0 \vdash A$.
In particular, RCA_0 and EL_0 have the same consistency strength.

Characterizing WKL

Theorem (Essentially by [1])

The following are equivalent over EL_0

1. WKL
2. $\Pi^0_1\text{-}\mathrm{AC}^\vee + \Sigma^0_1\text{-}\mathrm{DML}$
    - $\Pi^0_1\text{-}\mathrm{AC}^\vee$:
      $\forall x(A(x) \vee B(x)) \to \exists \alpha \forall x((\alpha(x) = 0 \to A(x)) \wedge (\alpha(x) \neq 0 \to B(x)))$
      for $A, B \in \Pi^0_1$
    - $\Sigma^0_1\text{-}\mathrm{DML}$: $\neg(A \wedge B) \to (\neg A \vee \neg B)$ for $A, B \in \Sigma^0_1$

De Morgan's law

- $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
  - $\surd$ $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$
  - $\surd$ $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$
- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
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- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
  - $\neg(A \wedge B) \to \neg A \vee \neg B \leftarrow$ does not hold in EL_0
  - √ $\neg A \vee \neg B \to \neg(A \wedge B)$

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  - $\surd$ $\neg A \vee \neg B \to \neg(A \wedge B)$

Some generalization

- $\forall x(\neg(\exists i < x)(A(i)) \leftrightarrow (\forall i < x)(\neg A(i)))$
  - $\surd$ $\forall x(\neg(\exists i < x)A(i) \to (\forall i < x)\neg A(i))$
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Schemata we consider

For a class Γ (Σ^0_k or Π^0_k) of formulae, we consider the following schemata:

- Γ-DML: $\neg(A \wedge B) \to \neg A \vee \neg B$, for $A, B \in \Gamma$
- Γ-GDML: $\forall x(\neg(\forall i < x)A(i) \to (\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$
- Γ-WGDML: $\forall x(\neg(\forall i < x)A(i) \to \neg\neg(\exists i < x)\neg A(i))$ for $A(i) \in \Gamma$

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Easy observation

Over EL_0, the following holds:

1. $\Gamma\text{-GDML}$ yields $\Gamma\text{-DML}$
2. $\Gamma\text{-GDML}$ yields $\Gamma\text{-WGDML}$

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Variation of WKL

- WKL: Every infinite binary tree has a path.
- $\mathrm{WKL!!}$: Every infinite binary tree T which has at most one paths, i.e., if there are two paths then they are identical, has a path.
- $\mathrm{dn\text{-}WKL}$: If T is an infinite binary tree, then $\neg\neg(T$ has a path$)$

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Observation

$\mathrm{WKL} \Rightarrow \mathrm{WKL!!} \Rightarrow \mathrm{dnWKL}$ ([5])

Something I know so far

Let $\Delta(\Gamma)$ be the smallest class containing Γ and closed under $\wedge, \vee, \rightarrow, \neg, \forall x < t, \exists x < t$

In the presence of an appropriate induction

- $\mathrm{EL}_0 + \Delta(\Sigma^0_k)\text{-IND} \vdash \Sigma^0_k\text{-DML} \Leftrightarrow \Sigma^0_k\text{-GDML}$
- $\mathrm{EL}_0 + \Delta(\Pi^0_k)\text{-IND} \vdash \Pi^0_k\text{-DML} \Leftrightarrow \Pi^0_k\text{-GDML}$
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In the presence of WKL variants or some choice principles

- $\mathrm{EL}_0 + \mathrm{WKL} \vdash \Sigma^0_1\text{-GDML}$
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Σ^0_k and Π^0_k

- $\mathrm{EL}_0 + \Sigma^0_1\text{-DML} \vdash \Pi^0_1\text{-DML}$ ([4])
- $\mathrm{EL}_0 + \Sigma^0_1\text{-GDML} \vdash \Pi^0_1\text{-GDML}$ (Kawai)
- $\mathrm{EL}_0 + \Sigma^0_{n+1}\text{-GDML} + \Sigma^0_n\text{-DNE} + \Sigma^0_n\text{-IND} \vdash \Pi^0_{n+1}\text{-GDML}$

Some tips for Γ-WGDML

Lemma

Over EL_0, the following are equivalent.

1. Σ^0_k-WGDML
2. $\forall x((\forall i < x)\neg\neg A(i) \to \neg\neg(\forall i < x)A(i))$, for any Σ^0_k formula $A(i)$.

Proof. Let $A(i) \equiv \exists j B(i,j)$ be a Σ^0_1 formula.

$$
\begin{aligned}
& \neg(\forall i < x)\exists j B(i,j) \to \neg\neg(\exists i < x)\neg\exists j B(i,j) \\
\iff & \neg(\exists i < x)\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j) \\
\iff & (\forall i < x)\neg\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j).
\end{aligned}
$$

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$$\begin{aligned} & \neg(\forall i < x)\exists j B(i,j) \to \neg\neg(\exists i < x)\neg\exists j B(i,j) \\ \iff & \neg(\exists i < x)\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j) \\ \iff & (\forall i < x)\neg\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j). \end{aligned}$$

Observation

2 is a special case of the following Σ^0_k-DNS (*Double Negation Shift*), which does not hold in EL_0:

$$\forall i\neg\neg A(i) \to \neg\neg\forall i A(i), \text{for any } A(i) \in \Sigma^0_k$$

Facts

- $\mathrm{EL}_0 \nvdash \Pi^0_1\text{-IND}$ ([2]), $\mathrm{EL}^*_0 + \Pi^0_1\text{-IND} \nvdash \Sigma^0_1\text{-IND}$ ([7]).
- $\mathrm{EL}_0 + \Pi^0_1\text{-IND}$ proves $\neg\neg\Sigma^0_1\text{-IND}$ ([7]), i.e., for each $A(x) \in \Sigma^0_1$,
$$\neg\neg A(0) \wedge \forall x(\neg\neg A(x) \to \neg\neg A(x+1)) \to \forall x \neg\neg A(x),$$
- EL_0 proves, for each $A(i,j) \in \Sigma^0_1$,
$$\forall x((\forall i)\exists j A(i,j) \leftrightarrow \exists y(\forall i < x)(\forall j < y)A(i,j))$$
- $\mathrm{EL}_0 + \Pi^0_1\text{-IND}$ proves, for $A(i,j) \in \Sigma^0_1$,
$$\forall x((\forall i < x)\neg\neg\exists j A(i,j) \leftrightarrow \neg\neg\exists n(\forall i < x)(\forall j < n)A(i,j))$$

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Theorem

$\mathrm{EL}_0 + \Pi^0_1\text{-IND}$ proves $\Sigma^0_1\text{-WGDML}$.

(Proof)
$$\neg(\forall i < x)\exists j B(i,j) \to \neg\neg(\exists i < x)\neg\exists j B(i,j)$$
$$\iff \neg(\exists i < x)\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j)$$
$$\iff (\forall i < x)\neg\neg\exists j B(i,j) \to \neg\neg(\forall i < x)\exists j B(i,j)$$
$$\underset{\Pi^0_1\text{-IND}}{\iff} \underbrace{(\forall i < x)\neg\neg\exists j B(i,j) \to \neg\neg\exists y(\forall i < x)(\exists j < y)B(i,j)}_{\Pi^0_1\text{-IND yields this}}$$

Induction, DML and GDML

Theorem

$\mathrm{EL}_0 + \Delta(\Sigma^0_k)\text{-IND} + \Sigma^0_k\text{-DML} \vdash \Sigma^0_k\text{-GDML}$

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(Idea for the proof)

Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Delta^0_0$.

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg\exists j A(0,j) \wedge (\underline{\forall i(0 < i < x \to \exists j A(i,j))}$

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- Underlined part is equivalent to some Σ^0_1 formula.
- Hence, Σ^0_1-DML, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$

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- If the former is the case, repeat this process.

Induction, DML and GDML

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(Idea for the proof)

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- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg\exists j A(0,j) \wedge \underline{(\forall i(0 < i < x \to \exists j A(i,j))}$
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- Hence, $\Sigma^0_1\text{-DML}$, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

Induction, DML and GDML

Theorem

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- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

Where we need $\Delta(\Sigma^0_k)$-IND?

Bounded comprehension: Using induction, take
$u = \langle u_0, ..., u_{x-1}\rangle \in \{0,1,2\}^*$ s.t.

- $u_{x-i-1} = 0 \to \neg\exists j(i,y)$,
- $u_{x-i-1} = 1 \to \neg(\forall k < x-i-1)\exists y A(k,y)$, and
- $u_{x-i-1} = 2 \to (\exists k < i) u_k = 0$

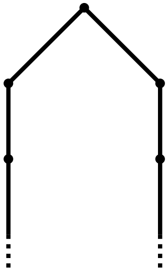
WKL and Σ^0_1-DML (LLPO)

Fact ([1])

$\mathrm{EL}_0 + \mathrm{WKL} \vdash \Sigma^0_1\text{-DML}$

(Idea for the proof)
Assume $\neg(\exists x A(x) \wedge \exists B(x))$, where $A(x), B(x) \in \Delta^0_0$.
Consider the following tree T:

$$T = \{u \in \{0,1\}^* : (u_0 = 0 \to \neg A(|u|)) \wedge (u_0 = 1 \to \neg B(|u|))\}$$



Since T must have a branch of any length, T has a path α.
$\alpha(0) = 0$ implies $\neg\exists x A(x)$ and $\alpha(0) = 1$ implies $\neg\exists x B(x)$.

WKL and GDML

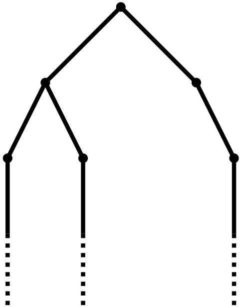
Theorem

$\mathrm{EL}_0 + \mathrm{WKL} \vdash \Sigma^0_1\text{-GDML}$

(Idea for the proof)
Assume $\neg(\forall i < x)\exists j A(i,j)$ for $A(i,j) \in \Delta^0_0$. Consider the following T:

$$T = \{1^i 0^j : i < x, \neg A(i,j)\}$$

The case of $x = 3$



Since T must have a branch of any length, T has a path α.
Find the least $i < x$ s.t. $\alpha(i) = 1$. Then $\neg\exists j A(i,j)$.

WKL and GDML

Theorem

$\mathrm{EL}_0 + \mathrm{WKL} \vdash \Sigma^0_1\text{-GDML}$

(Idea for the proof)

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg\exists j A(0,j) \wedge (\underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ^0_1 formula.
- Hence, Σ^0_1-DML, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

How to repeat the process?

WKL and GDML

Theorem

$\mathrm{EL}_0 + \mathrm{WKL} \vdash \Sigma^0_1\text{-GDML}$

(Idea for the proof)

- $\neg(\forall i < x)\exists j A(i,j)$ implies $\neg\exists j A(0,j) \wedge (\underline{\forall i(0 < i < x \to \exists j A(i,j))})$
- Underlined part is equivalent to some Σ^0_1 formula.
- Hence, $\Sigma^0_1\text{-DML}$, we have $\neg\exists j A(0,j)$ or $\neg\forall i(0 < i < x \to \exists j A(i,j))$
- If the former is the case, repeat this process.
- At some $i < x$, we must have $\neg\exists j A(i,j)$.

How to repeat the process?

WKL implies the following choice principle:

- $\Pi^0_1\text{-AC}^\vee$:
  $\forall x(A(x) \vee B(x)) \to \exists\alpha\forall x((\alpha(x) = 0 \to A(x)) \wedge (\alpha(x) \neq 0 \to B(x)))$
  for $A, B \in \Pi^0_1$

By this choice principle, we can take the right direction at once.

Some observations

- Constructive reverse mathematics aims to characterize mathematical principles with choice principles (existence of functions), logical principles and sometime induction principles.
- Choice, logical, and induction principles are independent at a glance.
  - Realizability model: Full of choice principle, but weak induction, no non-logical principle
  - Total recursive function model: Full induction, but weak choice, no non-logical principle
  - Classical non-standard model: Classical logic, but weak choice, weak induction
- De Morgan's law (DML) is considered as a logical principle.
- But DML is generalized by induction or choice principle.
- How choice, logical and induction principles affect each other?

References

1. J. Berger, H. Ishihara and P.Schuster, *The weak König's lemma, Brouwer's fan theorem and de Morgan's law, and dependence choice*, Reports on mathematical logic 47 (2012), 63–86 DOI:10.4467/20842589RM.12.003.0684
2. S. Buss, *Intuitionistic validity in T-normal Kripke structures*, Ann. Pure Appl. Logic 59, 1993 pp. 159-173
3. M. Fujiwara and T. Nemoto *On the decomposition of WKL!!*, Phil. Trans. R. Soc. A 381: 20220010., 2023, `https://doi.org/10.1098/rsta.2022.0010`
4. H. Ishihara, *Markov's principle, Church's thesis and Lindelöf's Theorem*, Indagationes Mathematicae Volume 4, Issue 3,1993, pp.321-325
5. J. R. Moschovakis, *Another Unique Weak König ' s Lemma WKL!!*, In: For Helmut Schwichtenberg, with respect and appreciation for his encouragement and friendship, Logic, Construction, Computation, edited by Ulrich Berger, Hannes Diener, Peter Schuster and Monika Seisenberger, Berlin, Boston: De Gruyter, 2012, pp. 343-352. `https://doi.org/10.1515/9783110324921.343`
6. T. Nemoto and K. Sato, *A marriage of Brouwer's Intuitionism and Hilbert's Finitism I: Arithmetic*, The Journal of Symbolic Logic, 87(2), pp.437-497, 2022 doi:10.1017/jsl.2018.6
7. K. Wehmeier, *Fragments of HA based on Σ_1-induction*, Arch. Math. Logic (1997) 37: pp.37–49