Intuitionistic logic and proof theory

Constructive Mathematics: Foundations and Practice,
In memory of Errett Bishop
Niš, June 2023

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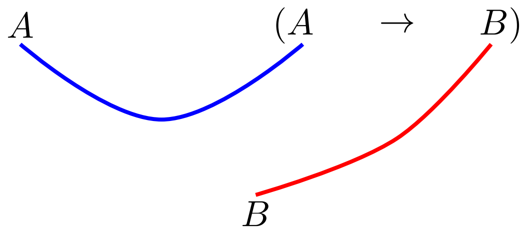

This talk is dedicated to my professor, Kosta Došen 1954-2017.

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Constructive Mathematics

There is no unique Constructive mathematics.

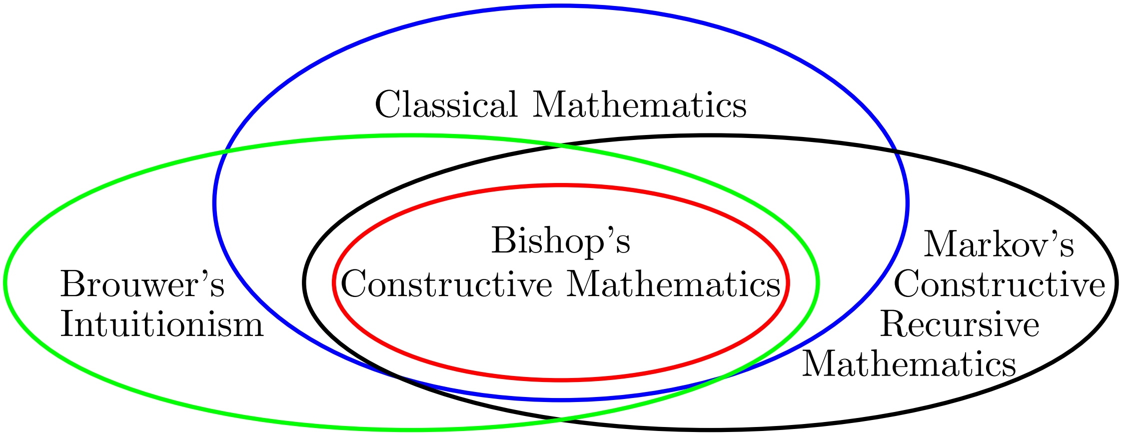


FIGURE 1. Troelstra's taxonomy of Constructive Marthematics

Intuitionistic Logic

However, intuitionistic logic underlies Bishop's Constructive Mathematics, Intuitionism and Constructive Recursive Mathematics. The following classically valid formulae are not valid in intuitionistic logic.

$$A \vee \neg A$$

$$\neg\neg A \to A$$

$$\neg\exists x \neg A(x) \to \forall x A(x)$$

N.B. Read $A(x)$ as "if $x\colon D^2 \to D^2$ is a continuous function, then it has a fixed point".

Intuitionistic Logic

First principles of intuitionistic logic were formulated by Brouwer (although he was not on the side of formalization of mathematics). Kolmogorov (1925) and Heyting (1930) started to study intuitionistic logic in a formal way. Moreover, they introduced independently an informal interpretation named *Brouwer-Heyting-Kolmogorov* (BHK) *interpretation*, which serves as a standard explanation of intuitionistic logic.

In this interpretation, a meaning of a statement is given by an explanation what should be its proof. For example, a proof of $A \to B$ is a construction (function) which transforms any proof of A into a proof of B, or a proof of $\exists x A(x)$ is a pair (a, q), where a belongs to the domain and q is a proof of $A(a)$.

Semantics of Intuitionistic Logic

Intuitionistic logic has several very interesting formal semantics like for example Kripke semantics (derived from Kripke semantics for modal logics), topological semantics and algebraic semantics. One may consider each of these semantics to generalize the one preceding it.

There is an intuitionistic proof of the completeness result for intuitionistic first-order logic with respect to algebraic semantics. Also, there is an intuitionistic proof of the completeness result for intuitionistic propositional logic with respect to all these semantics.

However, this talk is devoted to the proof theory of intuitionistic logic.

Gentzen's Sequent System LJ

Main concepts: *sequents* $\Gamma \vdash \Theta$, where Γ is a sequence of formulae, and Θ is either empty or contains a single formula.

Axiom schema: $A \vdash A$

Structural rules: Thinning, Contraction, Interchange, Cut

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Lambda}$$

Operational rules: For connectives and quantifiers (left and right introduction)

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash \Lambda}{A \to B, \Gamma, \Delta \vdash \Lambda} \qquad \frac{A(a), \Gamma \vdash \Theta}{\exists x A(x) \Gamma \vdash \Theta}$$

*the *eigenvariable* a does not occur in the lower sequent

Variants of LJ

Proof-theoretical analysis of intuitionistic logic includes formation of systems adequate for proving intuitionistic unprovability or provability of some propositional or predicate formulae. Such are systems developed by Kleene and later by Negri and von Plato.

Axiom schema: $A, \Gamma \vdash A$

Structural rules are eliminated.

Operational rules: For connectives and quantifiers (left and right introduction)

$$\frac{A \to B, \Gamma \vdash A \quad B, A \to B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta} \qquad \frac{A(a), \exists x A(x), \Gamma \vdash \Theta}{\exists x A(x), \Gamma \vdash \Theta}$$

Again, this is not the main subject of our talk.

General vs. Reductive Proof Theory

General proof theory is a field of mathematics inaugurated by Prawitz in the beginning of the 1970s. Its foundations are in Gentzen's thesis. The *reductive proof theory* (the name also coined by Prawitz) is devoted to consistency proofs for formalized fragments of mathematics, and is also based on the work of Gentzen.

One of the topics in general proof theory (according to Prawitz) is the following:
2.3 The representation of proofs by formal derivations. In the same way as one asks when two formulas define the same set or two sentences express the same proposition, one asks when two derivations represent the same proof; in other words, one asks for identity criteria for proofs or for a "synonymity" (or equivalence) relation between derivations.

The Normalization Conjecture

In early 1970s, Prawitz formulated a so-called *normalization conjecture* tied to *natural deduction* system for intuitionistic logic. A derivation in a natural deduction system is in *normal form* when it contains no *maximum formula* and no *maximum segment*. An example of a maximum formula is the following.

$$\dfrac{\dfrac{\mathcal{D}_1}{A} \qquad \dfrac{\dfrac{\overset{[A]}{\mathcal{D}_2}}{B}}{A \to B}}{B} \qquad\qquad\qquad \begin{array}{c} \dfrac{\mathcal{D}_1}{A} \\ \dfrac{\mathcal{D}_2}{B} \end{array}$$

The left-hand side derivation reduces to the right-hand side derivation. The *equivalence* of derivations is the reflexive, transitive and symmetric closure of the immediate reducibility relation.

Prawitz's Normalization Conjecture. Two derivations represent the same proof if and only if they are equivalent.

Curry-Howard Correspondence

The Curry-Howard correspondence acts at three different levels. The formulae correspond to types, the derivations correspond to terms and the normalization of derivations corresponds to reductions in typed calculi.

This correspondence may serve as a justification of normalization conjecture. Especially, since we have the following result (a typed version of Böhm's Theorem) proved independently by Statman, Simpson and Došen&P.

Theorem. *If we add a new equality to the simply-typed λ-calculus, then every equality is provable.*

Curry-Howard Correspondence

Example. Let $t_1\colon A$ be a λ-term of type A and for $x\colon A$ and $t_2(x)\colon B$ consider the term $\lambda_x.t_2\colon A \to B$. The terms corresponding to derivations from the previous slide are the following.

$$\dfrac{\dfrac{\mathcal{D}_1}{A} \qquad \dfrac{\begin{matrix}[A]\\ \dfrac{\mathcal{D}_2}{B}\end{matrix}}{A \to B}}{(\lambda_x.t_2(x))t_1\colon B} \qquad\qquad \dfrac{\begin{matrix}\dfrac{\mathcal{D}_1}{A}\\ \mathcal{D}_2\end{matrix}}{t_2(t_1)\colon B}$$

The right-hand side term is obtained by β-reduction applied to the term corresponding to the left-hand side derivation.

What is a proof?

Došen: It may be asked whether the Normalization Conjecture, or another answer to the question "When are two representations of proofs equal?", brings us closer to answering the basic question "What is a proof?". If our answers to the first question about identity criteria for proofs cut across various representations of proofs, so that when one representation is translated into another they agree about the notion of identity of proof, it seems that we could answer the basic question "What is a proof?" by taking one of our representations and declaring that the equivalence classes of this representation determine the notion we are seeking.

The Generality Conjecture

A *generalization* of a derivation in a propositional logic consists of diversifying the propositional letters without changing the rules of inference.

Generality Conjecture. Two derivations represent the same proof if and only if they have the same generalization.

Example. One can derive p from $p \wedge p$ either by using the rule

$$\frac{A \wedge B}{A} \quad \text{or} \quad \frac{A \wedge B}{B}.$$

Hence, the generalizations of $p \wedge p \vdash p$ are $p \wedge q \vdash p$ and $p \wedge q \vdash q$.

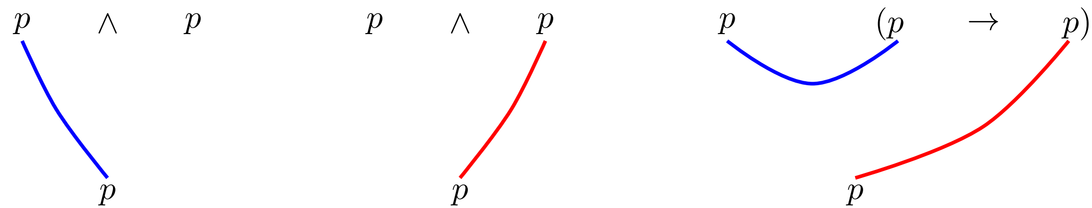
We will formalize this conjecture within the field called *Categorial Proof Theory.*

Categorial Proof Theory

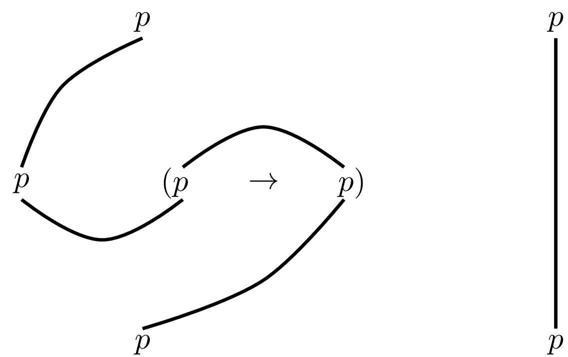
In late 1960s, with a series of papers Lambek introduced the categorial proof theory. The main idea is to turn a sequent system into a category—the formulae are objects, the sequents are arrows, the axiomatic sequents are the identity arrows and the cut rule corresponds to composition. In this way, different logics correspond to different types of categories like for example symmetric monoidal categories (intuitionistic linear logic), cartesian and bicartesian closed categories (intuitionistic propositional logic). The intuitionistic logic is much more suitable for this approach than the classical logic. The difficulties with classical logic come from the fact that under some natural assumptions, the underlying categories are trivial (preorders).

Cobordisms

The generalization of derivations could be easily followed by "diagrams".



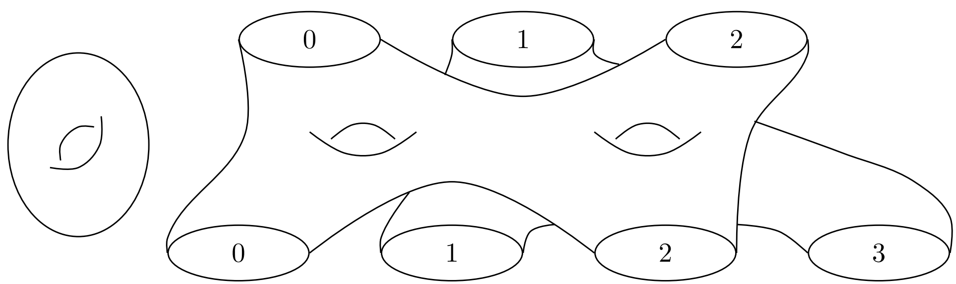
The composition of diagrams is given by pasting.



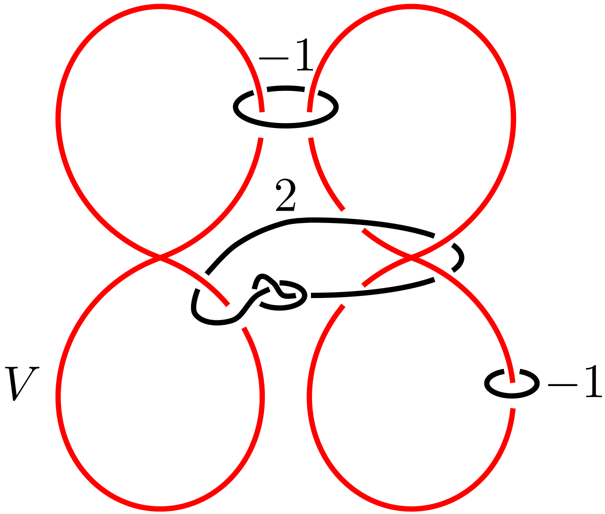
Cobordisms

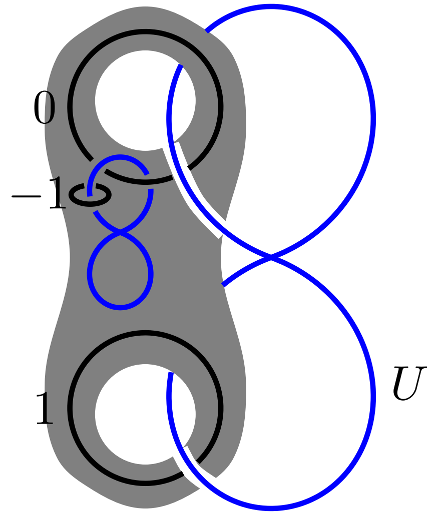
Our diagrams from the preceding slide are nothing but 1-dimensional cobordisms. The generality conjecture says that such a cobordisms govern the equality of derivations. Is it possible to use higher-dimensional cobordisms for such purposes?

Our current investigation is devoted to this question. Increasing dimension by 1 is interesting and important for the proof theory of modalities, but 3-dimensional cobordisms have much richer structure.

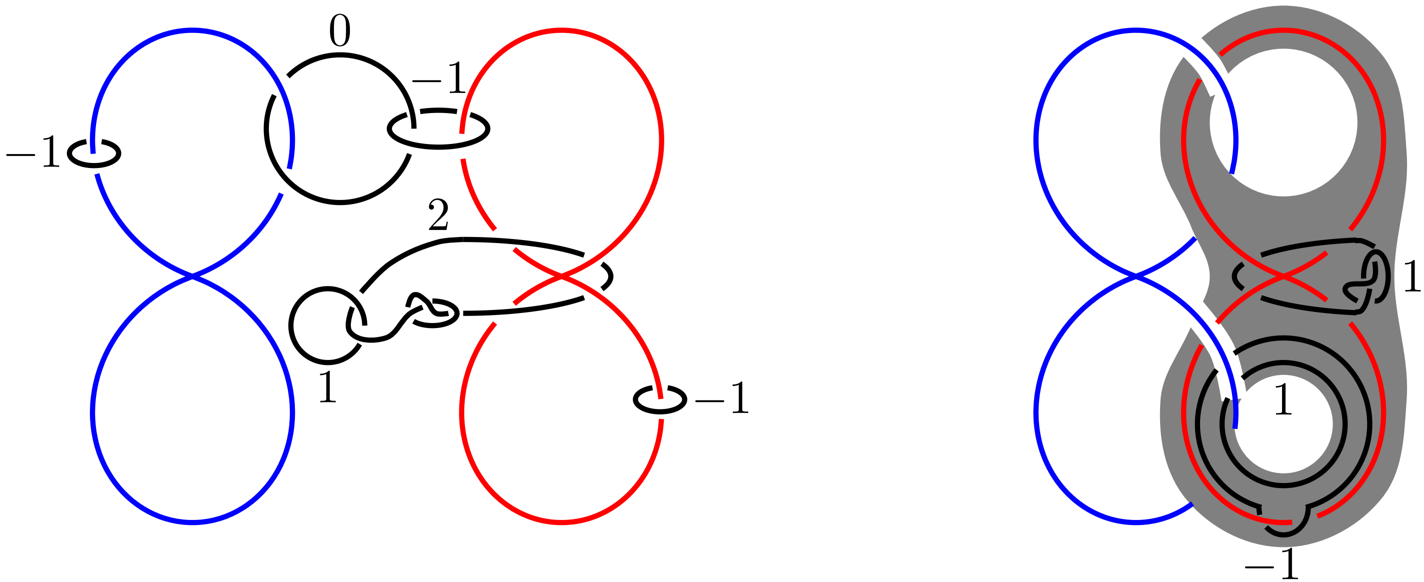


Cobordisms

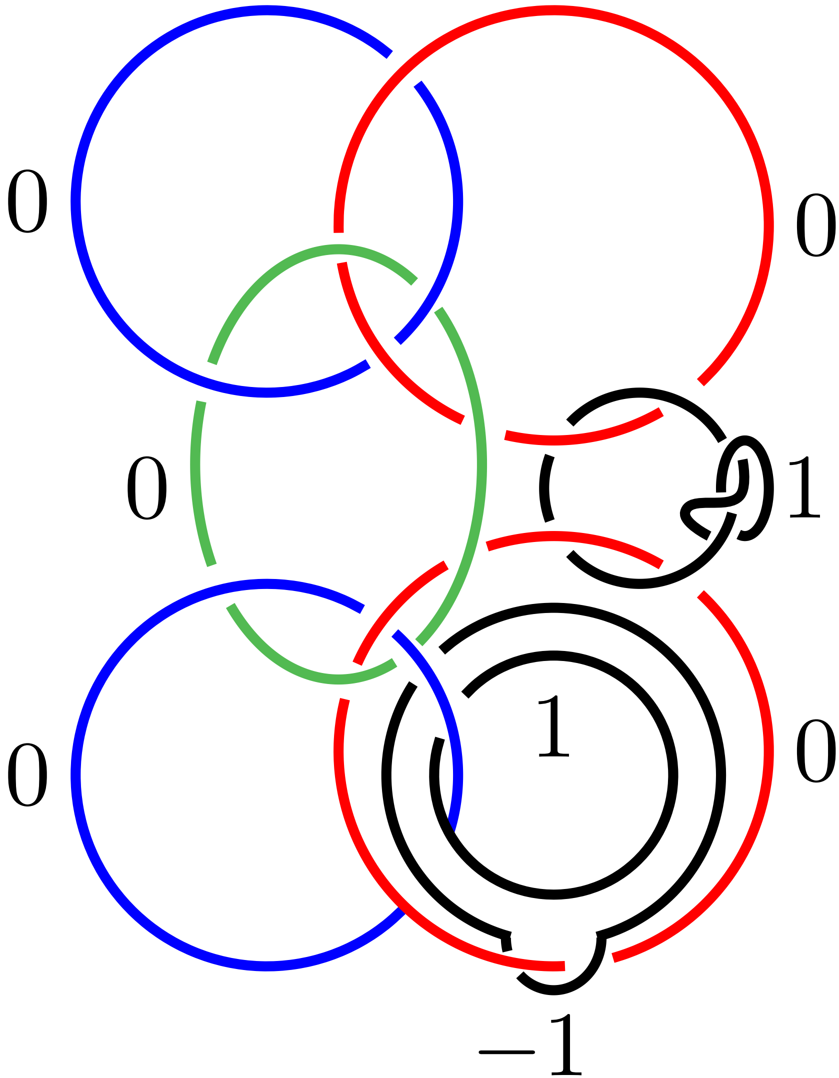




Cobordisms



Cobordisms



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