Posterior analysis of Gompertz distribution based on records

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- Record values are seen as those observations that overpass the previous ones in a sequence of observations
- They were introduced by Chandler [1] as a framework that deals with the time of occurrence of weather extremes from a sequence of weather conditions which can be interpreted as realizations of independent and identically distributed (iid) random variables
- It has been acknowledged that records have a major importance in sports, economics, meteorology, medicine and so on

kth record values

Introduction and definitions

- Let $T_{1,k} = k,\ R_{1(k)} = X_{1:k}$ and for $n \geq 2$, let $T_{n,k} = \min\{j : j > T_{n-1,k},\ X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\}$, where $X_{i:m}$ denotes the ith order statistics in a sample of size m from the underlying iid sequence $\{X_i,\ i \geq 1\}$
- The sequence $\{T_{n,k},\ k \geq 1\}$ is denoted as sequence of kth upper record times
- The sequence $\{R_{n(k)} = X_{T_{n-1,k}-k+1:T_{n-1,k}},\ n \geq 1\}$ is denoted as sequence of upper kth record values ([2])
- Basically, an upper kth record value is the kth largest yet seen in a partial sample
- For case $k = 1$, kth records reduce to ordinary records
- Record statistics have found their place in various statistical fields such as characterization problems, goodness of fit tests, predictions, information theory, reliability analysis, etc.

Gompertz distribution
Properties

- Cumulative distribution function(cdf)

$$F(x;\alpha,\beta) = 1 - e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)},\ x > 0,\ \alpha > 0,\ \beta > 0$$

- Probability density function(pdf)

$$f(x;\alpha,\beta) = \beta e^{\alpha x-\frac{\beta}{\alpha}(e^{\alpha x}-1)},\ x > 0,\ \alpha > 0,\ \beta > 0$$

- Fits mortality tables and tumor growth data
- Generalization of exponential distribution
- Model highly negatively skewed data in survival analysis

Bayesian inference

Choice of a prior distribution

- Prior distributions are an essential part in the formulation of posterior distribution
- Select a prior distribution that better represents the dependence structure of the parameters regarding the information
- Making an optimal decision based on researchers beliefs - subjective priors
- On contrary, objective priors are used
- Many noninformativie priors have been proposed over the years
- Our goal is on the Jefferys priors, reference priors, Zellner priors and probability matching priors

Problem under investigation

- If both parameters are unknown, specifying a joint prior which support lies on the positive real axis may lead to computational complexities ([3])
- A discretization of priors has been proposed as a solution
- Complicated form of the posteriors and practically useless.
- We dealt with this issue through objective priors
- The basic idea is to determine is the obtained posterior proper

Theorem 1.

(a) The Jeffrey's prior for (α, β) is

$$\pi_J(\alpha, \beta) \propto \alpha^{-1}\beta^{-1/2}.$$

(b) The posterior distribution under $\pi_J(\alpha, \beta)$ is improper.

Theorem 2.

(a) If α is the parameter of interest and β is the nuisance parameter, the reference prior is of the form

$$\pi_{R1}(\alpha, \beta) \propto \alpha^{-1}\beta^{-1},$$

and, if β is the parameter of interest and α is the nuisance parameter, the reference prior for (α, β) is

$$\pi_{R2}(\alpha, \beta) \propto \alpha^{-3/2}\beta^{-1/4}.$$

(b) The posterior distribution under the reference prior π_{R1} and π_{R2} is improper.

Theorem 3.

(a) The MDI prior for the parameters (α, β) is given by

$$\pi_{MDI}(\alpha, \beta) \propto \frac{e^{\alpha}}{\beta}.$$

(b) The posterior distribution under $\pi_{MDI}(\alpha, \beta)$ is proper.

Theorem 4.

(a) When β is the parameter of interest and α is the nuisance parameter, the second-order probability matching prior has the form of

$$\pi_{M_1}(\alpha, \beta) \propto F_1(\alpha) \cdot G_1(\beta), \quad (1)$$

where $F_1(\alpha) \propto e^{(1+c_1)(\alpha)}$ and $G_1(\beta) \propto \beta^{c_1}$, and c_1 as an arbitrary constant.

(b) For the case when $c_1 = -1$, the posterior distribution under π_{M_1} is proper.

Data analysis

- Record sample from Gompertz distribuiton with parameters $\alpha = 1.483$ and $\beta = 5.572$ [4]
- Ordinary records, i.e. $k = 1$
- Sample: 0.12528, 0.21211, 0.22784, 0.26063, 0.65258, 0.66056, 0.68255, 0.79385, 0.83778, 0.92206
- Posteriors are generated by the random-walk Metropolis algorithm implemented in the package *MHadaptive* found in software R [5].

Table
Subtitle

Table: Summary of the Bayesian estimates.

Prior	Parameter	Median	SD	95% HDI
π_{MDI}	α	2.8798	1.3664	(0.0843, 5.396)
	β	2.0434	2.6282	(0.0877, 7.8284)
π_{M_1}	α	1.4186	0.9927	(0.0001, 3.3829)
	β	4.8633	3.2196	(0.5117, 11.6949)

Plots of estimates

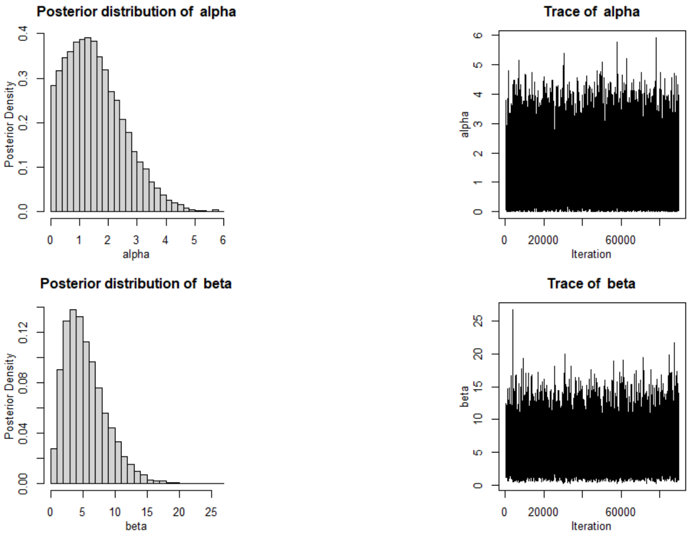

Figure: Posterior estimates of the parameters α and β under prior π_{M_1}.

Further research

- Generating procedure for kth records from Gompertz distribution
- Simulation study for comparison on the performance of the posteriors by invoking different priors

References

[1] K. N. Chandler. The distribution and frequency of record values. *Journal of the Royal Statistical Society Series B (Methodological)*, 14, 1952.

[2] W. Dziubdziela and B. Kopociński. Limiting properties of the k-th record values. *Applicationes Mathematicae*, 2(15):187–190, 1976.

[3] L. Wang, Y. Shi, and W. Yan. Inference for Gompertz distribution under records. *Journal of systems engineering and electronics*, 27(1):271–278, 2016.

[4] Z. F. Jaheen. A Bayesian analysis of record statistics from the Gompertz model. *Applied Mathematics and Computation*, 145(2-3):307–320, 2003.

[5] R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2015.

Thank you for your attention!